Let's begin with a very gentle introduction to the foundations of topology, and by the end, we'll use topology to show that there are infinitely many prime numbers. First, let's pick our set, or our universe that we want to work in. We really only talk about integer primes, so consider ***Z*,** the set of all integers. From here on out, everything I talk about will be about ***Z***. Topology is very concerned with studying subsets and their properties. So, we want to think about subsets of ***Z*** and how they work.

First, we need to think about infinite arithmetic sequences. The name might be intimidating, but we all know what these are. {…, -7, -4, -1, 2, 5, 8, 11, 14, …} is an example of an infinite arithmetic sequence. It's a list of numbers where the difference between two consecutive numbers is always the same, and there are infinitely many numbers in the list. In the above example, you add 3 to get the next number, and of course you keep subtracting 3 to get the negative terms in the sequence. ***Z*** itself is also an infinite arithmetic sequence, because to get the next number, just add 1.

Now, back to topology. There are these rules called topological axioms that tell you whether or not subsets of ***Z*** are "nice." I won't explain what these axioms are to make things easier to understand, but just know that if subsets follow these axioms, then we can work with them. For example, let's say I want to think about subsets of ***Z*** that only contain prime numbers. So, I want to think about sets such as {2, 5, 13} and {3, 5, 7, 11, 13, 31}, but I'm not interested in sets such as {-10, 3, 6}. If I want to see whether these subsets are "nice" subsets that I can work with, then I need to check that they satisfy the topological axioms. That's how the axioms work.

So, what you do is you define which subsets of ***Z*** you want to think about (for example, subsets of **Z** which only contain negative numbers, or subsets of ***Z*** that contain the number 8, etc.) and you check if they satisfy the axioms or not. If a collection of subsets doesn't satisfy the axioms, then you can't work with them, and you have to think about some other subsets. If the collection of subsets does satisfy the axioms, then you call these subsets **open**. So, if you have a subset, which I'll call *X*, and you say that *X* is open, what you're saying is that *X* belongs to some collection of subsets that you're thinking about, and this collection satisfies the axioms.

I know, it's pretty abstract. If you're a bit confused, reading the last two paragraphs a couple more times should help. I encourage you to not have doubts before you continue. And don't worry about what the topological axioms are. I'll say which collections of subsets satisfy the axioms and which don't without explaining why, and you'll have to trust me. (Although, if you're particularly interested, you can look up what the axioms are!)

Now, we're ready to explore the ideas of Furstenburg's proof. Remember, we're thinking about ***Z*** and subsets of ***Z*.** What we want to do is call a subset of ***Z*** an open set if it's an infinite arithmetic sequence or a union of multiple infinite arithmetic sequences. For example, we say {…, -7, -4, -1, 2, 5, 8, 11, 14, …} is an open set, because it's an infinite arithmetic sequence. ***Z*** itself is an open set, because ***Z*** is an infinite arithmetic sequence. The set {…, -12, -10, -8, -5, -4, 0, 4, 5, 8, 10, 12, …} is open, because it is exactly the union of the sets {…, -12, -8, -4, 0, 4, 8, 12, …} and {…, -10, -5, 0, 5, 10, …}, and both of these are infinite arithmetic sequences.

However, {…, -3, -1, 1, 3, 5, 6, 7, 9, 11, …} is not an open set. It contains all of the odd numbers, which is an infinite arithmetic sequence, but it randomly has the number 6 in there, and 6 isn't a part of any infinite arithmetic sequence in that set. So that set is not open. We also are going to consider the empty set to be an open set, but that's not too important, so don't worry much about it.

One thing to take note of is that a non-empty finite set can't be open, because a finite set can't contain any infinite arithmetic sequences. The finite set {-4, -2, 0, 2, 4, 6, 8, 10} looks like it should be an open set, but it's not. It's an arithmetic sequence, but it's not an *infinite* arithmetic sequence, so by our definition of what an open set is, that set isn't open. We'll use the fact that a finite set can't be open later.

Remember that when we say a set is open, we're saying that its collection satisfies the topological axioms. You'll have to trust me that when we define open sets in this way (again, we say a set is open if it's an infinite arithmetic sequence or a union of multiple arithmetic sequences), the axioms are satisfied. In other words, the collection of all subsets of $\boldsymbol{Z}$ that are infinite arithmetic sequences or unions of them satisfies the axioms. When Furstenburg wrote his proof, he had to, of course, prove that this collection satisfies the axioms, but we won't here.

I'm repeating myself a lot, but it's worth giving a quick summary to make sure we're all on the same page. An infinite arithmetic sequence is an infinite sequence of integers where the difference between any two consecutive integers is the same. We're thinking about subsets of $\boldsymbol{Z}$ that are infinite arithmetic sequences or a union of them. The collection of all of these subsets is a collection that satisfies the topological axioms. Because the collection satisfies the axioms, any subset in the collection is called an open set.

We've talked about sets that are open, so you might be wondering if there's such a thing as a closed set. Yes, there is! But, it's not what you think it is. First, let's recall what the complement of a set is. Intuitively, the complement of a set A is everything except A, and will be denoted A^C. But remember, here, everything is happening in $\boldsymbol{Z}$. So for us, the complement of the odd integers is the even integers. The complement of {-3, 2, 5} is {…, -6, -5, -4, -2, -1, 0, 1, 3, 4, 6, 7, 8, …}, and so on.

Now, we're ready to talk about closed sets. We say that a set is **closed** if its complement is an open set. And, as you'd expect, a set is **not closed** if its complement is not open. It's a strange definition, I know. Shortly, I'll give lots of examples of closed and open sets to make the concept more clear. One thing to note is that "closed" is not the same thing as "not open." We've seen what it means for a set to be open, and now we see what it means for a set to be closed. But it's possible for a set to be both open and closed. It's also possible for a set to be neither open nor closed! Let's do some examples to make more sense of this.

Remember, to prove that there are infinitely many primes, we say that a subset of $\boldsymbol{Z}$ is open if it's an infinite arithmetic sequence or the union of multiple arithmetic sequences. We saw earlier that the set {…, -12, -10, -8, -5, -4, 0, 4, 5, 8, 10, 12, …} is *open*, because it's the union of the infinite arithmetic sequences {…, -12, -8, -4, 0, 4, 8, 12, …} and {…, -10, -5, 0, 5, 10, …}. So what about the set A = {…, -11, -9, -7, -6, -3, -2, -1, 1, 2, 3, 6, 7, 9, 11, …}? If you notice,

A^C = {…, -12, -10, -8, -5, -4, 0, 4, 5, 8, 10, 12, …}, which is open, as we just saw. Therefore, *A* is *closed* because its complement is open.

As another example, consider the set {1, 2, 3, 4}. This set is *not open*, because it's finite, so it doesn't contain an infinite arithmetic sequence. So, consider the set *B* = {…, -3, -2, -1, 0, 5, 6, 7, …}. Notice that B^C = {1, 2, 3, 4}, which is not open, as we just saw. Therefore, *B* is *not closed*, because its complement is not open.

So far, we've described some very basic topological concepts, such as open and closed sets. In order to make these concepts more clear, we've seen some examples of open, closed, not open, and not closed sets using the definition of "open" that Furstenburg used in his proof. Remember, you can define open sets however you want as long as they follow the axioms! And of course, the notion of being an infinite arithmetic sequence or a union of them is what Furstenburg decided to call "open."

To give more perspective on where we're going, I'll mention now that Furstenburg's proof is a proof by contradiction. We're going to suppose that there are finitely many primes, and then, using concepts such as open and closed sets, arrive at a contradiction. Here's how the proof is going to go: we're going to consider a specific subset of ***Z*,** which for now I'll call *D*, and we're going to write *D* in two different ways. If we write the *D* one way, we'll show that the *D* is closed. If we choose to write *D* in the other way, we'll show that *D* is not closed. A set can't be both closed and not closed, so this is a contradiction.

Before we get to the punch line, there are two crucial observations we need to make. The first observation is that if *X* is a non-empty subset of ***Z***, and *X* is finite, X^C is not closed. Why? Remember how we mentioned that if *X* is finite, *X* cannot be open? That was because all open sets, by our definition, are infinite arithmetic sequences or a union of multiple infinite arithmetic sequences. So a finite set surely can't satisfy this condition because all open sets are infinite. So, a finite set *X* is not open. Again, recall that a set is closed if its complement is open. Therefore, because *X* is not open, X^C is not closed.

The second observation has to do with sets that are both open and closed. Remember how we mentioned that it is possible for a set to be both open and closed? It turns out that an infinite arithmetic sequence is both open and closed! We've said many times that an infinite arithmetic sequence is an open set because that's what we defined open sets to be! But now we need to see why it's closed as well.

Consider the set *H* = {…, -9, -6, -3, 0, 3, 6, 9, …}. By our definition of open sets, *H* is open. But what is the complement of *H*? H^C = {…, -8, -7, -5, -4, -2, -1, 1, 2, 4, 5, 7, 8, …}. But notice that H^C is exactly equal to the union of {…, -8, -5, -2, 1, 4, 7, …} and {…, -7, -4, -1, 2, 5, 8, …}. Notice that both of these are infinite arithmetic sequences! So, H^C is a union of infinite arithmetic sequences, so by our definition, H^C is open. Therefore, by the definition of a closed set, *H* is closed, because H^C is open. So, we have shown that *H* is both open and closed! This isn't a proof, but it should convince you that an infinite arithmetic sequence is both an open and a closed set. If you're not convinced, you're encouraged to work out more examples.

There is one last fact we need, but I won't prove that it's true because it has to do with the topological axioms. You'll have to take my word that it's correct. If $A_1, A_2, \ldots, A_n$ are all closed sets, then the union of all of them is a closed set. So, the union of a finite number of closed sets is a closed set. This is a general fact that is easily proven if one knows the topological axioms.

Now, we're ready to describe Furstenburg's proof. One last time, let's recap what we know about topology, so everything is fresh in our minds when we try to understand the proof. We said an open set is an infinite arithmetic sequence or a union of multiple infinite arithmetic sequences (and we'll also say the empty set is open). Under this definition, the topological axioms are satisfied, so we actually are allowed to say that these sets are open. A set is closed if its complement is open. A non-empty, finite set cannot be open, because all non-empty open sets are infinite. Thus, the complement of a non-empty finite set cannot be closed. We also showed that in addition to being open by definition, an infinite arithmetic sequence is a closed set as well. Finally, we needed to mention that a union of a finite number of closed sets is closed. That's everything we need. Here is Furstenburg's proof of the infinitude of primes.

Consider the set $D = \boldsymbol{Z} \setminus \{-1, 1\}$. In other words, D is the set of all integers except for 1 and -1. {-1, 1} is a non-empty, finite set. Therefore, D, which is the complement of {-1, 1} cannot be closed, because D is the complement of a non-empty finite set. So we conclude that D is not closed.

We can write D in another way. Suppose by contradiction that there are finitely many primes. Let's assume that there are n primes in total. By the Fundamental Theorem of Arithmetic, every integer except -1 and 1 can be written as a nonzero product of primes. So, consider the infinite arithmetic sequences that contain 0 and where you count by each prime number. So, for 2, which is a prime, consider the infinite arithmetic sequence $A_1 = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$, and I call it A_1 because 2 is the first prime. For the prime 3, consider the $A_2 = \{-9, -6, -3, 0, 3, 6, 9, \ldots\}$, and it's denoted A_2 because 3 is the second prime. Following the same pattern, $A_3 = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}$, and so on. Do this for all n primes, creating the sets $A_1, A_2, \ldots, A_n$.

The Fundamental Theorem of Arithmetic guarantees that every number except for -1 and 1 appears on one of these lists. If an integer is a multiple of 3, it will appear in A_2. If an integer is a multiple of 2, it will appear in A_1. The key point is that because we know every integer except 1 and -1 is either prime or has some prime factors, every integer except 1 and -1 is going to be somewhere in the collection of sets $A_1, A_2, \ldots, A_n$. We assumed that there are n primes, so the sets $A_1, A_2, \ldots, A_n$ represent the infinite arithmetic sequences containing all of the n prime numbers. The union of all of these, as in, the union of $A_1, A_2, \ldots, A_n$, contains every integer except 1 and -1.

But this is exactly the set D! Remember that D is the set of all integers except 1 and -1. Thus, D is exactly equal to the union of $A_1, A_2, \ldots, A_n$. We showed that an infinite arithmetic sequence is actually a closed set. So, A_1 is closed, A_2 is closed, …, and A_n is closed. We also know that the union of a finite number of closed sets is closed. So, the union of $A_1, A_2, \ldots, A_n$, which is D, is closed. Therefore, we conclude that D is closed.

We have shown that under the assumption that there are finitely many primes, the set $D = \boldsymbol{Z} \setminus \{-1, 1\}$ is both closed and not closed. This is a contradiction, so there are infinitely many primes.